

CREEP RUPTURE

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INTRODUCTION

Enoch Thulin, Swedish pioneer in aeronautics, was killed in an airplane accident on a routine flight in May, 1919, now more than 43 years ago. In those days this was no uncommon fate for a pilot, particularly so a test pilot. Figure 1 shows Dr. Thulin, in whose memory this lecture is being delivered. He lived to be 38 years of age. Besides being his own test pilot, he was a pioneer in scientific aeronautics, a gifted designer of planes and motors, and the founder of an airplane industry with well over 800 employees.

The sources of knowledge about Thulin's activities are rather scanty. My own memories of him do not go beyond those of the man in the street. You must remember that at the time of his death I was about to finish my freshman year as a mechanical engineering student at the Royal Institute of Technology. Thulin's intimate friend and collaborator, Dr. Ivar Malmer, gave his first lectures in aeronautical engineering here a few years later, 1922 or so. But Ivar Malmer, too, has been dead for many years and I have been compelled to scan written sources in order to obtain any information at all about the Thulin airplane industry at Landskrona in southern Sweden.

Thulin made it his goal to increase flight safety—an urgent task indeed in his time. He worked along several lines of thought. He had an aerodynamic laboratory modeled on the Eiffel plant in France, but with several original contributions of his own. An original inventor and designer of a rotating internal combustion engine, Thulin had realized that the dependability of his motor was, above all, a question of knowledge about the materials used. Figure 2 shows his materials laboratory.¹ It was surprisingly well outfitted; the balance-type testing machine, the Brinell hardness tester, the Charpy pendulum, and the rotating-beam fatigue machine were exactly the same as those of the Royal Institute when I took over the materials laboratory there in 1936. The only addition there consisted of eight creep testing machines, for in the meantime the phenomenon of creep had made its appearance among the chores to be attended to by designers and material specialists in mechanical engineering and aeronautics.



Fig. 1

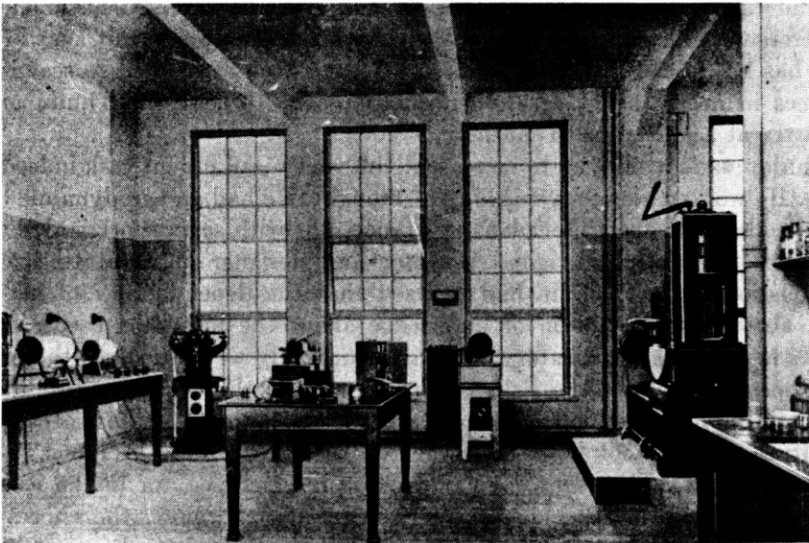


Fig. 2

This brings us back to the topic of today's lecture. The material problem of contemporary aeronautics and space flight is, of course, much more varied than it was in 1919 and shows aspects entirely unknown at that time. This is due to the fact that today we fly so much faster and so much higher than we did 50 years ago. Aerodynamic heating was then negligible. Also, today the working temperatures of motors and jet engines are far above those of Thulin's day.

Out of the new problems, I have picked that of creep rupture for reasons of my own. I shall try to show that the essential features of this phenomenon could be derived from a few fundamental properties inherent to most structural metals in a certain temperature range. Rupture stress and elongation as a function of strain rate will be proved to have a continuous transition from the lowest speed in creep to the highest speeds encountered, for example, in shocks. I shall start with a short review of some physical facts.

May these introductory words serve as a link between Enoch Thulin's time and the present.

PHYSICAL BACKGROUND

Metal physicists teach us today that fracture of metal structures nucleates from local plastic deformation. This is true for all types of fracture—ductile, brittle, or fatigue. In all these cases we have to take into consideration plastic deformation prior to the ultimate formation of a crack. Such deformation may be localized to a very small part of the crystal grains or may be spread out over the entire structure. In either case, deformation may be understood only by taking into account the motion of crystal defects known as dislocations.

The simplest case of a dislocation may be seen in Fig. 3; dislocations in this form were introduced by Sir Geoffrey Taylor in 1934. In the middle of the structure a vacancy is seen moving horizontally from left to right under the action of a shear stress much lower than would have been needed if the whole upper part of the structure had moved one atomic spacing simultaneously.

The notion of dislocations was introduced as a hypothesis in the 1930's by several scientists independently in order to explain the fact that crystals do deform plastically at a much lower stress than would be required if the crystal lattice were perfect. Today dislocations of the form just indicated, as well as of many more complicated ones, have been made visible and have even been filmed. They are to be considered a physical reality as much as the molecules of a gas. In general, it is necessary to use special artifices or the high magnifications

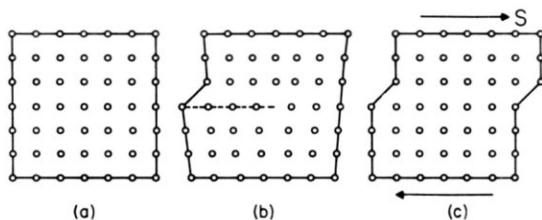


Fig. 3.

obtainable only with an electron microscope in order to reveal this strange material behavior (see, for example, Fig. 4, taken from Ref. 2).

A further study of metal plasticity in relation to the motion of dislocations shows that a great variety of mechanisms is possible. Each material shows its individual characteristics. A day may come when plastic behavior of structural metals can be predicted if their atomic structure is known. At present, I regret to say, the designer and computing engineer have little help from dislocation theory as far as calculation methods are concerned. At the same time, it is strange to note that the general global behavior of structural metals with regard to deformation and strength is very much the same for all. Certain simple fundamental "laws" may be stated. It is true that materials may behave differently with respect to ductility and brittleness. But still these differences are relative rather than absolute.

Out of the multiplicity of observations made on metals under stress I shall single out three, *viz.*,

- (a) the law of plastic deformation with strain hardening
- (b) the law of viscous flow under constant stress in the so-called secondary stage of creep
- (c) the law of deterioration of the material with time under action of stress.

These three laws pertain to the phenomenological behavior of the material. Their explanation from the point of view of dislocation theory may be very intricate indeed and show individual characteristics for different materials. Their utility may be proved only by their simplicity and—of course—by experience.

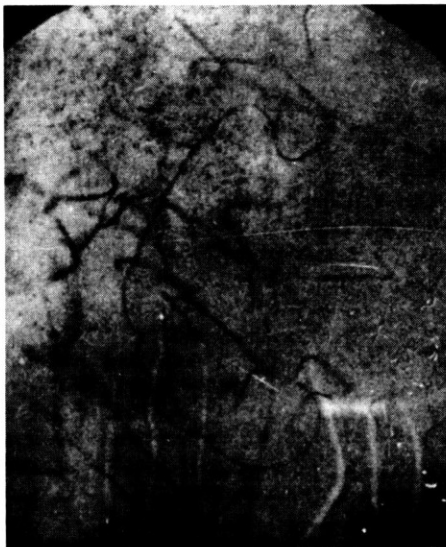


Fig. 4. Curvilinear dislocations found in specimen deformed at a slow rate at about 1000°C.

PHENOMENOLOGICAL APPROACH

In what follows we shall consider nonrecoverable plastic deformation only. We shall also neglect the accompanying elastic deformations.

(A) PLASTIC DEFORMATION WITH STRAIN-HARDENING

The plastic strain ϵ under the action of a stress σ may be expressed in the form

$$\epsilon = \left(\frac{\sigma}{\sigma_0} \right)^{n_0} \quad (1)$$

where σ_0 and n_0 are material constants. Equations of this form for the description of the results of ordinary tension tests were proposed already in the nineteenth century—e.g., by C. Bach in Germany and somewhat later by P. Ludwik.³ More recent data to support Eq. (1) have been collected by J. D. Lubahn and R. P. Felgar.⁴ The plastic strain as computed from Eq. (1) neglects velocity effects and is by definition unrecoverable. If unloading takes place from a stress value $\sigma = \sigma_a$, then ϵ has to be kept constant until σ again exceeds the value σ_a , cf. Fig. 5.

(B) VISCOUS FLOW UNDER CONSTANT STRESS

Above a certain critical temperature each metal, when loaded under constant stress σ , shows plastic deformation—so-called creep—with strain ϵ , increasing progressively with time t as seen in Fig. 6 for aluminum, which is reproduced from J. Dorn.⁵ The terminology “primary,” “secondary” and “tertiary” stages of creep, shown in the Fig. 6, corresponds to decrease, constancy, and again increase of the strain rate $d\epsilon/dt$, and is due to the work of E. N. da C. Andrade early in this century. The strain rate during the secondary stage is strongly depending on the stress σ and also on the prevailing temperature as indicated

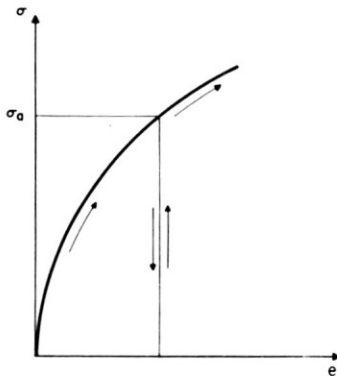


Fig. 5.

in Fig. 6. The stress dependence of $d\epsilon/dt$ was investigated in the 1920's very completely by F. N. Norton⁶ and presented in the form of a power law

$$\frac{d\epsilon}{dt} = \left(\frac{\sigma}{\sigma_c} \right)^n \quad (2)$$

which we shall, for shortness, call Norton's law. The quantities σ_c and n are material constants, depending on temperature. Many subsequent investigators have utilized equations of the type (2). It has been criticized, it is true, but recent investigations have established its validity, at least in the lower stress range. For moderate requirements and global considerations such as those of the present paper, it may be used as an interpolation formula over a wide range of stress.

(C) DETERIORATION WITH TIME UNDER STRESS

In the so-called tertiary stage of creep the material is known to become deteriorated. Voids are being formed due to coalescence of dislocations. Figure 7, reproduced from Ref. 7, shows the fracture appearance and also the influence of the ambient atmosphere. Considering a test piece in uniaxial tension under

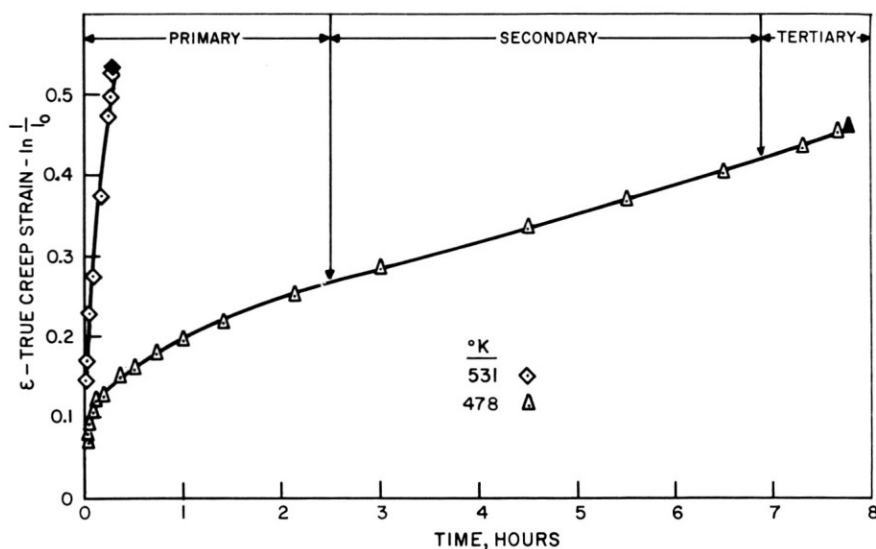


Fig. 6.

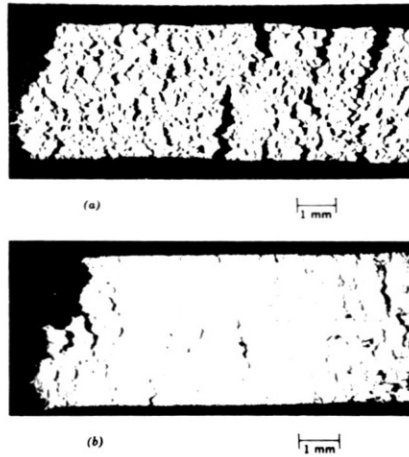


Fig. 7. (a) Vacuum-melted 80 nickel-20 chromium alloy containing 1.33% silicon tested at 1500° F. Rupture life, 870 hr; elongation, 45%. (b) Air-melted alloy also tested at 1500° F, 5,000 psi. Rupture life, 524 hr; elongation, 12%.

load P , only part A_r of the cross section A will support the load and we may define a damage factor

$$D = (A - A_r)/A$$

This means that the real stress σ_r on the supporting area A_r will be larger than the mean stress $\sigma = P/A$. It is reasonable to assume a relationship of the form

$$\frac{dD}{dt} = f(\sigma_r) = f\left(\frac{\sigma}{1 - D}\right) \tag{3}$$

for the increase of the damage D with time t . The stress σ may be applied at the time $t = 0$, when the material is in a virgin state and we have $D = 0$. When the damage is complete we have $D = 1$. Integration of the differential Eq. (3) will enable us to compute the time t_R to rupture. In order to achieve this we must know the function $f(x)$ of the variable x introduced in Eq. (3). If we add the assumption that the differential Eq. (3) be separable, this yields the simple form

$$f(x) = Cx^\nu \tag{4}$$

with C and ν material constants. Inserting Eq. (4) in Eq. (3) and integrating, we immediately obtain

$$C \int_0^{t_R} \sigma^\nu dt = \int_0^1 (1 - D)^\nu dD = \frac{1}{\nu + 1} \tag{5}$$

The condition (5) was introduced by L. M. Kachanov⁸ and utilized also for time-dependent stress σ . As shown by J. Hult and the present author,⁹ Kacha-

nov's theory is equivalent to the earlier "linear cumulative creep damage" law due to E. L. Robinson.¹⁰ If the stress $\sigma = \sigma_k = \text{constant}$, we obtain from Eq. (5)

$$C \sigma_k^\nu t_k = \frac{1}{\nu + 1} \quad (6)$$

and thus we may write Eq. (5) in the form

$$\int_0^{t_R} \left(\frac{\sigma}{\sigma_k} \right)^\nu dt = t_k \quad (7)$$

The material constants appearing in Eqs. (1), (2) and (7) may be given a form, more familiar to engineers. In fact, they are closely connected with well-known material properties encountered in the engineering literature. Thus we may write

$$\sigma_0^{n_0} = \frac{(\sigma_{0.2})^{n_0}}{2 \cdot 10^{-3}} \quad (8)$$

$$\sigma_c^n = \sigma_{c7}^n \cdot 10^7 \quad (9)$$

$$t_k \sigma_k^\nu = (\sigma_{cB}^{(5)})^\nu \cdot 10^5 \quad (10)$$

where $\sigma_{0.2}$ is the "proof stress" for 0.2 percent permanent set, σ_{c7} the "limiting creep stress" causing 1 percent creep strain in 10^5 hours and $\sigma_{cB}^{(5)}$ the stress causing creep rupture in 10^5 hours. The constants n_0 , n and ν will be shown in the sequel to fulfill certain fundamental relations.

The three laws (A), (B), and (C) may be combined in various ways.

We may, for example, calculate the total creep strain for time-dependent stress σ in combining Eqs. (1) and (2) and neglect, as before, strains of the order of elastic strains. We thus obtain

$$\frac{d\epsilon}{dt} = \frac{d}{dt} \left(\frac{\sigma}{\sigma_0} \right)^{n_0} + \left(\frac{\sigma}{\sigma_c} \right)^n \quad (11)$$

Here the irreversibility condition requires that the first term on the right side should be omitted if unloading takes place and then be left out of consideration until the stress reaches again the absolute value σ from which unloading took place. Equation (11) was proposed by the present author in 1952.¹¹ It has been further developed in a series of papers.¹² It is capable of representing the data of ordinary creep curves at constant stress in the form

$$\epsilon = \epsilon^{(0)} + \nu t \quad (12)$$

where

$$\epsilon^{(0)} = \left(\frac{\sigma}{\sigma_0} \right)^{n_0} \quad (13)$$

$$v = \left(\frac{\sigma}{\sigma_c} \right)^n \tag{14}$$

i.e., Eq. (11) may be expected to give a realistic representation of creep phenomena but for the first stages of primary creep in the shaded region of Fig. 8.

GENERALIZATION TO MULTIAXIAL STRESS

So far we have considered uniaxial stress only. The designer and calculating engineer needs formulae for multiaxial stresses here as in the classical theory of elasticity. Before entering our main subject we shall first generalize the previous considerations to multiaxial states of stress. Equations (1), (2), and (7) may be generalized to general states of combined stress, if we take account of certain additional facts supplied by experience. These may be in the simplest case

- I. Isotropy
- II. Incompressibility
- III. Independence of hydrostatic pressure

The state of stress may be represented by the principal stresses $\sigma_1, \sigma_2,$ and σ_3 . The corresponding principal strains may be $\epsilon_1, \epsilon_2,$ and ϵ_3 . The generalization will be stated for Eq. (11) as this equation comprises Eqs. (1) and (2) as special cases.

Isotropy requires expressions for the creep rates

$$v_1 = \frac{d\epsilon_1}{dt} \quad v_2 = \frac{d\epsilon_2}{dt} \quad v_3 = \frac{d\epsilon_3}{dt}$$

as symmetric functions of the quantities $\sigma_1, \sigma_2, \sigma_3$.

Incompressibility may be expressed in different ways, depending upon the geometry of the problem under consideration. In the case of small strain rates it is expressed simply

$$v_1 + v_2 + v_3 = 0 \tag{15}$$

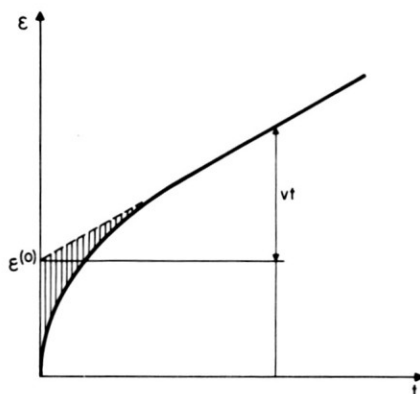


Fig. 8.

In order to make the creep rates v_i independent of hydrostatic pressure we prefer to use the stress deviation s_i instead of the stress σ_i , viz.,

$$s_i = \sigma_i - \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}, \quad i = 1, 2, 3 \quad (16)$$

Then we obviously have

$$s_1 + s_2 + s_3 = 0 \quad (17)$$

Further we introduce σ_e , the "effective stress," as a symmetric function of the deviation components s_i defined by

$$\begin{aligned} \sigma_e^2 &= \frac{3}{2} (s_1^2 + s_2^2 + s_3^2) \\ &= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 \end{aligned} \quad (18)$$

Obviously σ_e will itself be independent of a superimposed hydrostatic pressure. In the simple uniaxial case $\sigma_2 = \sigma_3 = 0$ we see that σ_e reduces to σ_1 .

After these preliminaries we may postulate as generalization of Eq. (11)

$$v_i = \frac{d}{dt} \left[G(\sigma_e) s_i \right] + F(\sigma_e) s_i, \quad i = 1, 2, 3 \quad (19)$$

The irreversibility condition here takes the following form: The first term on the right side of Eq. (19) shall be omitted whenever unloading takes place. This is to be understood in such a way that if we unload from a value $\sigma_e = \sigma_e'$ then the first term of Eq. (19) shall be left out of consideration until, on reloading, σ_e has again reached the value σ_e' .

Naturally, Eq. (19) shall reduce to Eq. (11) in the uniaxial case, say $\sigma_2 = \sigma_3 = 0$. This condition yields

$$G(\sigma_e) = \frac{3}{2} \left(\frac{\sigma_e}{\sigma_0} \right)^{n_0-1} \frac{1}{\sigma_0} \quad (20)$$

$$F(\sigma_e) = \frac{3}{2} \left(\frac{\sigma_e}{\sigma_c} \right)^{n-1} \frac{1}{\sigma_c} \quad (21)$$

The retention of the first term on the right side of Eq. (19) means taking account of irrecoverable plastic slip with the use of a "finite theory," thus violating the requirement that a theory of metal plasticity should by nature be "incremental." It also neglects the Bauschinger effect and other anisotropy effects of the flow. It neglects effect of anelasticity and of stress history. Its justification lies in the fact that in most cases we use this term as a correction term only, with the second term on the right side of Eq. (19) being the main term corresponding to viscous flow. Should occasionally the first term on the right side of Eq. (19) prevail over the second one, then in any case the ratio of the principal stresses shall remain at least approximately constant during the deformation.

Generalization of Eq. (7) to multiaxial states of stress may, according to Kachanov, be achieved by substituting for σ the actual value of the maximum principal stress $\sigma_{i \max}$ henceforth supposed to be a tension stress. The time to rupture t_R may then be determined from

$$\int_0^{t_R} \left(\frac{\sigma_{i \max}}{\sigma_k} \right)^v dt = t_k \tag{22}$$

In this form the theory has been successfully applied by Kachanov⁸ to cases of homogeneous stress distribution—e.g., to creep rupture of cylindrical, thin-walled tubes under internal pressure. Kachanov's hypothesis is supported by experiments on thin-walled tubular specimens of low-alloy cast steel and rolled copper in combined tension and torsion by A. E. Johnson and N. E. Frost.⁵

THEORIES OF CREEP RUPTURE BY N. J. HOFF AND BY L. M. KACHANOV

Creep rupture is the more or less brittle fracture that occurs at the end of the tertiary stage if a test piece—e.g., of a structural steel—is subjected to a constant tensile load at elevated temperature, say above 400°C. As already indicated, the physical background of this behavior may be complicated enough. Nevertheless, if for a creep rupture test, initial stress σ_{10} is plotted against lifetime t^* on a log/log basis, curves of very much the same general appearance will be obtained for a number of structural metals.

By way of example such typical curves are shown in Figs. 9 and 10, referring to two low alloy steels tested at 500°C, taken from a paper by K. Richard.¹⁴ Particularly in Fig. 10 the creep rupture curve is seen to be made up by two essentially straight lines with a smooth connection between them. For small t^* the slope is small and the fracture ductile, for larger t^* the slope is steeper and the fracture more or less brittle.

The first conscious attempt towards a phenomenological theory of creep rupture is due to N. J. Hoff (1953). He has given a full account of his results before the ICAS First Congress in Madrid, 1958.¹⁵

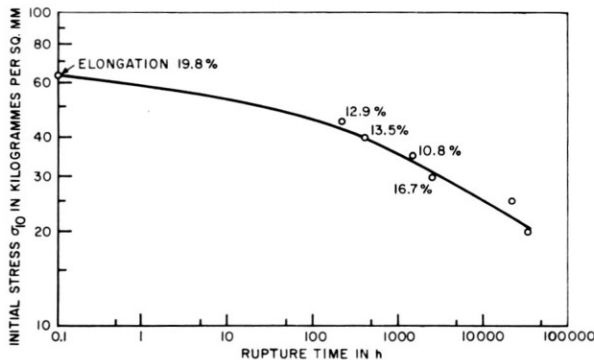


Fig. 9.

Considering a test piece in uniaxial tension with constant load P , Hoff assumes homogeneous conditions of stress and strain in a gage length l with cross section A , the initial values of these quantities at $t = 0$ being l_0 and A_0 respectively. Thus he obtains the condition of incompressibility in the form

$$A_0 l_0 = A l \quad (23)$$

The creep rate will be

$$\frac{d\epsilon}{dt} = v = \frac{1}{l} \frac{dl}{dt} = - \frac{1}{A} \frac{dA}{dt} \quad (24)$$

The stress will be $\sigma_1 = P/A$ and from Eq. (2) Hoff obtains, neglecting the effect of primary creep

$$v = - \frac{1}{A} \frac{dA}{dt} = \left(\frac{P}{A\sigma_c} \right)^n \quad (25)$$

Integration with satisfaction of the initial condition yields

$$n \left(\frac{P}{\sigma_c} \right)^n t = A_0^n - A^n \quad (26)$$

Thus the cross section will decrease from its initial value A_0 to $A = 0$ in the finite time t_H^* (subscript H for Hoff), where

$$t_H^* = \frac{A_0^n \sigma_c^n}{nP^n} = \frac{1}{n} \left(\frac{\sigma_c}{\sigma_{10}} \right)^n \quad (27)$$

The stress σ_1 will increase with time from its initial value $\sigma_{10} = P/A_0$ according to the equation

$$\begin{aligned} \sigma_1 &= \frac{A_0 \sigma_{10}}{A} = \sigma_{10} [1 - n(\sigma_{10}/\sigma_c)^n t]^{-1/n} \\ &= \sigma_{10} \left(1 - \frac{t}{t_H^*} \right)^{-1/n} \end{aligned} \quad (28)$$

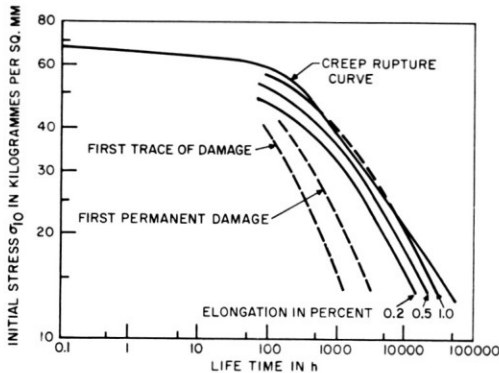


Fig. 10.

Thus σ_1 will become infinite when t approaches the value t_H^* . This entails large strains when approaching failure and would necessitate the use of a definition for finite strain. Apart from the simple uniaxial case, now under consideration, one should also take account of so-called "cross" terms in the constitutive equation, e.g., in Eq. (5). However, when we are primarily interested in the determination of the rupture time t_H^* , large strains will occur only during the latest small fraction of t_H^* and thus the secondary effects as measured by the cross terms will influence t_H^* but slightly.

Hoff's Eq. (27) represents a straight line with the slope $1/n$ in a log/log diagram between σ_{10} and t_H^* . This straight line corresponds to the left portion of the creep rupture curve of Fig. 10. The right part of this curve corresponds to Kachanov's relation (6) and as the slope of this part is $1/\nu$ the general trend of creep-rupture curves indicates the following relation between n and ν

$$n > \nu \tag{29}$$

Kachanov has also given a theory⁸ for the smooth part connecting the two straight portions of the creep rupture curve. He assumes creep according to Eqs. (23) through (28) to operate simultaneously with deterioration according to Eq. (7). Then inserting Eq. (28) into Eq. (7), he obtains

$$t_k = \int_0^{t_R} \left(\frac{\sigma_{10}}{\sigma_k} \right)^\nu \left(1 - \frac{t}{t_H^*} \right)^{-\nu/n} dt = \frac{n}{n - \nu} \left(\frac{\sigma_{10}}{\sigma_k} \right)^\nu t_H^* \left[1 - \left(1 - t_K^*/t_H^* \right)^{n-\nu/n} \right] \tag{30}$$

where t_K^* (subscript K for Kachanov) denotes that particular value of t_R for which rupture occurs. This naturally requires

$$t_K^* < t_H^* \tag{31}$$

otherwise ductile fracture would occur according to (27). From Eq. (30) we obtain

$$t_K^* = t_H^* \left\{ 1 - \left[1 - \frac{n - \nu}{n} \left(\frac{\sigma_k}{\sigma_{10}} \right)^\nu \frac{t_k}{t_H^*} \right]^{n/n-\nu} \right\} \tag{32}$$

which proves that the inequality (31) will hold, whenever

$$1 - \frac{n - \nu}{n} \left(\frac{\sigma_k}{\sigma_{10}} \right)^\nu \frac{t_k}{t_H^*} \geq 0 \tag{33}$$

Introducing the notation

$$\bar{\sigma} = \left[\frac{\sigma_c^n}{(n - \nu) t_k \sigma_k^\nu} \right]^{1/n-\nu} \tag{34}$$

we then have from Eq. (32)

$$t_K^* = \frac{(\sigma_c/\sigma_{10})^n}{n} \left\{ 1 - \left[1 - \left(\frac{\sigma_{10}}{\bar{\sigma}} \right)^{n-\nu} \right]^{n/n-\nu} \right\} \tag{35}$$

valid for $\sigma_{10} < \bar{\sigma}$, in which case $t_R = t_K^*$. If on the other hand $\sigma_{10} > \bar{\sigma}$, we have $t_R = t_H^*$ as before, because in that case we would obtain $t_K^* > t_H^*$ from Eq.

(35). For small values of σ_{10} , we see from Eq. (35) that t_K^* will behave asymptotically as $\sigma_{10}^{-\nu}$, i.e., t_K^* will approach the straight line, Eq. (6), Fig. 11. The stress $\sigma_{10} = \bar{\sigma}$ forms the boundary between ductile and brittle fracture.

Thus Kachanov's combined theory of ductile and brittle fracture is able to represent the creep rupture curve of many structural metals in a large range of stresses.

In the case of nonhomogeneous stress distribution—e.g., the case of creep rupture of thick-walled cylindrical tubes under internal pressure—the problem becomes much more complicated.

According to Kachanov,¹³ it is then necessary to distinguish two different time periods $t = 0 \dots t_{R'}$ and $t = t_{R'} \dots t_{R''}$, until the final failure occurs at $t = t_{R''}$. During the first period $(0, t_{R'})$ local deterioration takes place and local failure begins at $t = t_{R'}$. A failure front will then be propagated through the material during the second period $(t_{R'}, t_{R''})$ and have penetrated the structural part in question at the time $t = t_{R''}$, when final failure occurs.

In the quoted paper Kachanov has made a series of beautiful applications of his theory.

POSSIBLE IMPROVEMENTS OF KACHANOV'S THEORY OF CREEP RUPTURE

If Kachanov's theory of combined ductile and brittle creep rupture as stated in the previous chapter be compared more carefully with experimental results, it will be seen that the slope of the ductile part of the curve as required by the theory in many cases is far too large. This may be seen in Fig. 10, where Norton's exponent n as calculated from the creep-rupture curve for a low-alloy steel at 500°C would amount to about 70, whereas the correct value would be of the order of, say, 7.

Deviations between experiments and theory in the ductile region of the creep rupture curve were noted already by Hoff.¹⁵ He makes certain attempts to improve the theory by taking account of the ultimate rupture stress and the primary creep as expressed with the "equation of state" suggested by A. Nadai. Similar calculations have also been made recently by V. S. Namestnikov.¹⁶

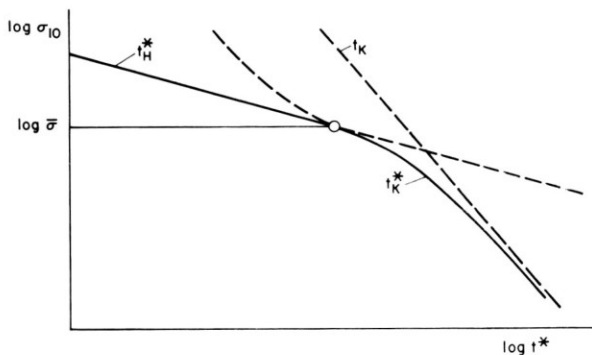


Fig. 11.

In these theories several new empirical constants are introduced and it should be possible to obtain better agreement with experiments by giving these constants proper values. However, these calculations are rather lengthy and the expressions for lifetime thus arrived at are very cumbersome indeed and are not easy to discuss.

Here shall now be made an attempt to take account of primary creep in the form introduced with Eq. (11). This equation shall be substituted for Eq. (2). Everything else remains unchanged. Instead of Eq. (25) we thus obtain

$$v = - \frac{l}{A} \frac{dA}{dt} = - \sigma_0^{-n_0} \left(\frac{P}{A} \right)^{n_0-1} \frac{P}{A^2} \frac{dA}{dt} + \sigma_c^{-n} \left(\frac{P}{A} \right)^n \quad (36)$$

Integration with the initial condition $A = A_0$ for $t = 0$ yields

$$A_0^n - A^n - \frac{n}{n - n_0} \sigma_0^{-n_0} (A_0^{n-n_0} - A^{n-n_0}) = n \left(\frac{P}{\sigma_c} \right)^n t \quad (37)$$

Putting as before $P/A = \sigma_1$, $P/A_0 = \sigma_{10}$ and assuming $A = 0$ for the particular value $t = t_p^*$ (subscript p for "primary") we obtain

$$t_p^* = \frac{(\sigma_c/\sigma_{10})^n}{n} \left[1 - \frac{n}{n - n_0} (\sigma_{10}/\sigma_0)^{n_0} \right] \quad (38)$$

for the time to ductile creep rupture. It is easily seen that the curve for t_p^* in a log/log plot falls below that of the straight line Eq. (27) for t_H^* . There is also a tendency of the curve for t_p^* to have a smaller slope as required by experience (see Figs. 9 and 10).

But it is also possible to modify Kachanov's combined theory in accordance with Eq. (11). Then combining Eqs. (7) and (36) we obtain

$$t_k = \left(\frac{\sigma_{10}}{\sigma_k} \right)^\nu A_0^\nu \int_0^{t_R} \frac{dt}{A^\nu} = \left(\frac{\sigma_{10}}{\sigma_k} \right)^\nu A_0^\nu \int_{A_0}^{\alpha_R A_0} \frac{dt}{dA} \frac{dA}{A^\nu} \quad (39)$$

where dt/dA is to be taken from Eq. (36) and where $A = \alpha_R A_0$ is the particular value of A for which brittle fracture will occur. This value corresponds to $t_R = t_{Kp}^*$ as computed from Eq. (37). We thus obtain the following parametric representation of t_{Kp}^* as a function of σ_{10} by means of the parameter α_R

$$t_k = \left(\frac{\sigma_{10}}{\sigma_k} \right)^\nu \left(\frac{\sigma_c}{\sigma_{10}} \right)^n \left\{ \frac{1}{n - \nu} (1 - \alpha_R^{n-\nu}) - \frac{(\sigma_{10}/\sigma_0)^{n_0}}{n - n_0 - \nu} (1 - \alpha_R^{n-n_0-\nu}) \right\} \quad (40)$$

$$t_{Kp}^* = \frac{(\sigma_c/\sigma_{10})^n}{n} \left\{ 1 - \alpha_R^n - \frac{n}{n - n_0} \left(\frac{\sigma_{10}}{\sigma_0} \right)^{n_0} (1 - \alpha_R^{n-n_0}) \right\} \quad (41)$$

Equations (40) and (41) are reduced to Eqs. (30) and (35) in the limiting case $\sigma_0 = \infty$. Transition from ductile fracture according to Eq. (38) to brittle fracture according to Eqs. (40) and (41) will now occur whenever

$$t_{Kp}^* < t_p^* \quad (42)$$

i.e., from Eqs. (40) and (41) as soon as

$$\alpha_R^n - \frac{n}{n - n_0} \left(\frac{\sigma_{10}}{\sigma_0} \right)^{n_0} \alpha_R^{n-n_0} \geq 0$$

or

$$\sigma_{10} \leq \alpha_R \sigma_0 \left(\frac{n - n_0}{n} \right)^{1/n_0} \quad (43)$$

On the other hand, from Eqs. (34) and (40) we have

$$\sigma_{10} = \bar{\sigma} \left\{ 1 - \alpha_R^{n-\nu} - \frac{n - \nu}{n - n_0 - \nu} \left(\frac{\sigma_{10}}{\sigma_0} \right)^{n_0} (1 - \alpha_R^{n-n_0-\nu}) \right\}^{1/n-\nu} \quad (44)$$

and from Eq. (43) we must have

$$1 \geq \alpha_R \geq \frac{\sigma_{10}}{\sigma_0} \left(\frac{n}{n - n_0} \right)^{1/n_0} \quad (45)$$

Inserting the minimum value of α_R according to Eq. (45) into Eq. (44), we obtain a condition for $\bar{\sigma}$ as a function of $\sigma_{10} = \bar{\sigma}_p$, the new boundary between ductile and brittle fracture, *viz.*,

$$\begin{aligned} \bar{\sigma} = \bar{\sigma}_p \left\{ 1 - \left(\frac{\bar{\sigma}}{\sigma_0} \right)^{n-\nu} \left(\frac{n}{n - n_0} \right)^{n-\nu/n_0} \right. \\ \left. - \frac{n - \nu}{n - n_0 - \nu} \left(\frac{\bar{\sigma}}{\sigma_0} \right)^{n_0} \left[1 - \left(\frac{\bar{\sigma}_p}{\sigma_0} \right)^{n-n_0-\nu} \left(\frac{n}{n - n_0} \right)^{n-n_0-\nu/n_0} \right] \right\}^{-(1/n-\nu)} \end{aligned} \quad (46)$$

In the limiting case $\sigma_0 = \infty$ we are reduced to $\bar{\sigma}_p = \bar{\sigma}$ as before in Fig. 11. For σ_0 finite this boundary will correspond to a slightly smaller value $\bar{\sigma}_p < \bar{\sigma}$. For $\sigma_{10} < \bar{\sigma}_p$ the rupture curve will be given by Eqs. (40) and (41) (brittle fracture), otherwise by Eq. (38) (ductile fracture). The asymptotic behavior of $t_{K_p}^*$ for small values of σ_0 will be seen from Eqs. (41) and (44). Equation (44) yields

$$\alpha_R = \left[1 - \left(\frac{\sigma_{10}}{\bar{\sigma}} \right)^{n-\nu} + \dots \right]^{1/n-\nu} = 1 - \frac{1}{n-\nu} (\sigma_{10}/\bar{\sigma})^{n-\nu} + \dots$$

valid for small values of σ_{10} . Inserting this in Eq. (41), we obtain

$$\begin{aligned} t_{K_p}^* &= \frac{(\sigma_c/\sigma_{10})^n}{n} \left\{ 1 - \left[1 - \frac{1}{n-\nu} (\sigma_{10}/\bar{\sigma})^{n-\nu} + \dots \right]^n + \dots \right\} \\ &= \frac{\sigma_c^n}{(n-\nu) \sigma_{10}^\nu \bar{\sigma}^{n-\nu}} \left\{ 1 + \dots \right\} \end{aligned} \quad (47)$$

and this equation proves that $t_{K_p}^*$ will behave asymptotically as $\sigma_{10}^{-\nu}$ as before. The general form of the rupture curve $t_{K_p}^*$ in the new combined theory will be

seen in Fig. 12. It offers better agreement with experiments on creep rupture than Kachanov's original theory. The new combined theory has been successfully applied to the creep rupture of thin-walled cylindrical tubes under internal pressure by the present author.¹⁷

An interesting observation may be made in connection with Eq. (38) for ductile creep rupture. In the case when n_0 approaches n we see that t_p^* would become infinite. As this is obviously against experience one may conclude

$$n_0 < n \tag{48}$$

In fact, this relation is in general agreement with the results of calculations of the author,¹⁸ based upon measurements by A. E. Johnson²⁰ in cases of multiaxial stresses. Should in some special case n_0 approach the limiting case $n_0 = n$, it follows from Eq. (38) that $t_p^* = \infty$, i.e., ductile fracture will not occur for such material. An example of such material is 0.17 percent cast steel at 450°C.¹⁸ Rupture time would then be determined from

$$t_R = \left(\frac{\sigma_c}{\sigma_{10}}\right)^n \frac{1}{n} \left[1 - \alpha_R^n + \left(\frac{\sigma_{10}}{\sigma_0}\right)^n \log \alpha_R \right] \tag{49}$$

where α_R is given by

$$t_k = \left(\frac{\sigma_{10}}{\sigma_k}\right)^\nu \left(\frac{\sigma_c}{\sigma_{10}}\right)^n \frac{1}{n - \nu} \left\{ 1 - \alpha_R^{n-\nu} - \left(\frac{\sigma_{10}}{\sigma_0}\right)^n \frac{n - \nu}{\nu} \cdot \frac{1 - \alpha_R^\nu}{\alpha_R^\nu} \right\} \tag{50}$$

We will return to the case $n = n_0$ for some comments in the following section.

DEFORMATION UNDER CONSTANT SPEED

In this concluding section we shall take a wider view of the phenomenon of creep rupture. We shall study the rupture stress in its dependence on strain rate in a wide range of this quantity.

Let us consider the ordinary tension test. A cylindrical test specimen is clamped between two grips and pulled apart by a central load. The two grips

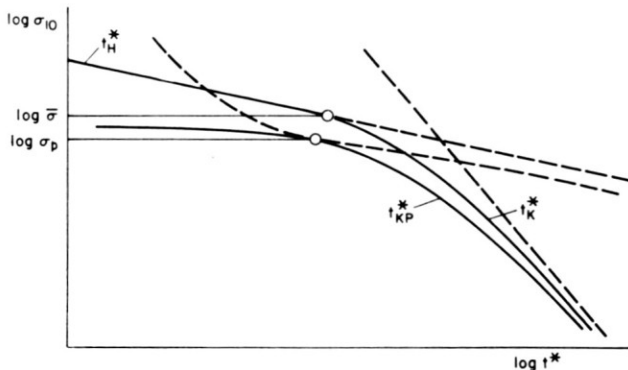


Fig. 12.

move with constant relative velocity u in the direction of the pull and the load, which should be carefully centered with respect to the specimen is adjusted so as to correspond to the velocity u at any moment.

If the gage length was l_0 at $t = 0$ it will be

$$l = l_0 + ut = l_0(1 + ut/l_0) = l_0(1 + v_0t) \quad (51)$$

at the arbitrary moment t , neglecting deformation of those parts of the test specimen that are outside the gage length. In Eq. (51) $v_0 = u/l_0$ is the strain rate at $t = 0$. For $t > 0$, the strain rate will decrease and amount to

$$v = \frac{1}{l} \frac{dl}{dt} = \frac{v_0}{1 + v_0t} \quad (52)$$

The total strain

$$\epsilon = \int_0^t v dt = \log(1 + v_0t) \quad (53)$$

thus giving rise automatically to the "natural strain" (P. Ludwik³).

Combining Eqs. (52) and (11), we obtain

$$\frac{d\epsilon}{dt} = \frac{v_0}{1 + v_0t} = \begin{cases} \frac{d}{dt} \left(\frac{\sigma}{\sigma_0} \right)^{n_0} + \left(\frac{\sigma}{\sigma_c} \right)^n & \text{if } \frac{d\sigma}{dt} > 0 \\ \left(\frac{\sigma}{\sigma_c} \right)^n & \text{if } \frac{d\sigma}{dt} \leq 0 \end{cases} \quad (54)$$

$$(55)$$

In this form we shall try to represent material behavior in a very large range of strain rates v_0 . The stress σ is supposed to be zero at time $t = 0$. Qualitatively, the time dependence of the quantities $d\epsilon/dt$, ϵ , and σ may be seen in Fig. 13.

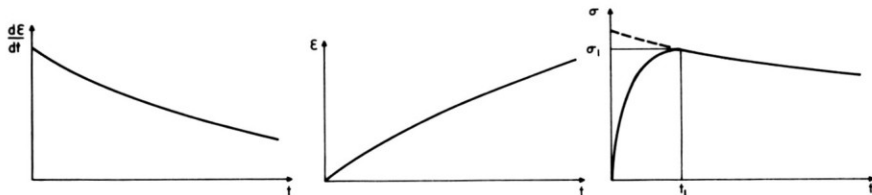


Fig. 13.

The only restriction† for such representation should be that the time t_1 for the stress to reach its maximum σ_1 shall be large as compared with the time for longitudinal stress waves to be propagated through the gage length, i.e., we must have

$$l_0/c_0 \ll t_1 \tag{56}$$

where c_0 is the wave velocity. The inequality, Eq. (56), excludes shock phenomena, such as stress waves and their interference, but not creep or rupture according to Eq. (7). The combination of Eqs. (54) or (55) with the rupture condition, Eq. (7), presents us with an intricate integration problem. I shall not bore you by giving any details, all the more as analytical solutions are possible only in special cases. It may suffice to show some limiting properties of the solutions of our equations.

In principle the rupture time t_R may come out greater or smaller than t_1 . In most practical cases, as pointed out by Kachanov, the quantity $\vartheta_R = v_0 t_R$, which coincides with the engineering "elongation," seldom exceeds 30 percent and is usually much less than so, irrespective of the value of v_0 . This greatly simplifies the calculations, as we may develop Eq. (53) in power series. Using the notation $v_0 t = \vartheta$, we obtain

$$\epsilon = \vartheta - \frac{\vartheta^2}{2} + \frac{\vartheta^3}{3} \dots \tag{57}$$

and here we need retain at most three terms of the development. This would then still leave us with an accuracy of better than 3 percent in the determination of ϵ . Introducing dimensionless quantities

$$\frac{\sigma}{\sigma_0} = \Sigma, \quad \frac{1}{v_0} \left(\frac{\sigma_0}{\sigma_c} \right)^n = \Omega, \quad t_k v_0 \left(\frac{\sigma_k}{\sigma_0} \right)^r = \theta \tag{58}$$

we may then write instead of Eqs. (54), (55), and (7)

$$\frac{1}{1 + \vartheta} = \begin{cases} \frac{d}{d\vartheta} \Sigma^{n_0} + \Omega \Sigma^n & \text{if } \frac{d\Sigma}{d\vartheta} > 0 \\ \Omega \Sigma^n & \text{if } \frac{d\Sigma}{d\vartheta} \leq 0 \end{cases} \tag{59}$$

$$\tag{60}$$

$$\theta = \int_0^{\vartheta_R} \Sigma^r d\vartheta \tag{61}$$

where $\vartheta_R = v_0 t_R$ is the engineering elongation, introduced above.

† Also, naturally, other parameters such as temperature must be kept constant.

The order of magnitude of the dimensionless quantities Ω and θ may be judged, utilizing Eqs. (8), (9), and (10) above, *viz.*,

$$\Omega = \frac{10^{-7}}{(2 \cdot 10^{-3})^{n/n_0}} \cdot \frac{1}{v_0} \left(\frac{\sigma_{0.2}}{\sigma_{c7}} \right)^n$$

$$\theta = 10^5 (2 \cdot 10^{-3})^{\nu/n_0} v_0 \left(\frac{\sigma_{cB}^{(5)}}{\sigma_{0.2}} \right)^\nu \quad (62)$$

Table 1 shows some typical numerical values of Ω and θ .

(A) THE CASE OF LOW VELOCITY

This case has been treated by Kachanov.¹³ He neglects influence of primary creep, which is the same as putting $t_1 = 0$. He thus may use Eq. (60) for the determination of the stress and obtains

$$\Sigma = \Omega^{-1/n} (1 + \vartheta)^{-1/n} \quad (63)$$

or introduced into Eq. (61)

$$\theta = \Omega^{-\nu/n} \int_0^{\vartheta_R} (1 + \vartheta)^{-\nu/n} d\vartheta = \Omega^{-\nu/n} \frac{n}{n - \nu} [(1 + \vartheta_R)^{n-\nu/n} - 1] \quad (64)$$

Assuming ϑ_R to be small he then may solve for ϑ_R and finally arrives at

$$\vartheta_R = \theta \Omega^{\nu/n} = t_k \left(\frac{\sigma_k}{\sigma_c} \right)^\nu v_0^{n-\nu/n} \quad (65)$$

which is, of course, independent of σ_0 . Kachanov states that experiments by A. V. Staniukovich are in general agreement with Eq. (65) as regards dependence on v_0 . Figure 14, reproduced from Ref. 19, however shows certain discrepancies.

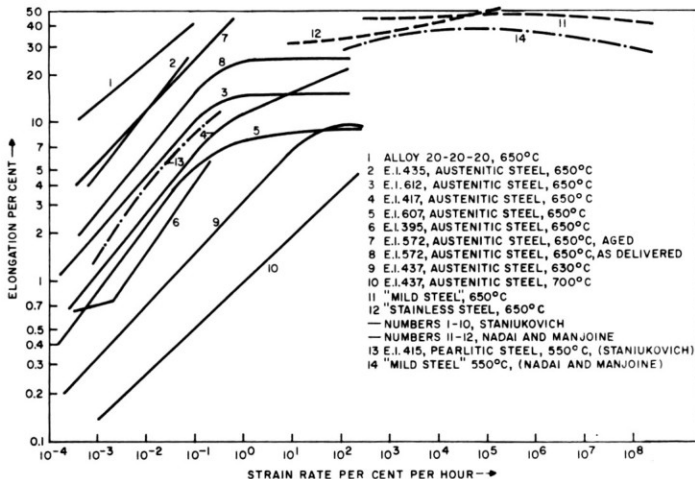


Fig. 14.

TABLE 1.
TYPICAL NUMERICAL VALUES OF Ω AND Θ

Number	r_0	Conditions Corresponding to					
		Creep Tests			High Speed Tension Tests		
		10^{-7}	10^{-4}	10^{-2}	1	10^3	1/h
1	$\sigma_{0.2}/\sigma_{cr} = 9/7$ $n_0 = 3$ $n = 5$ $\Omega = 1.1 \cdot 10^{-2}/r_0 =$ 2 4 5 5 $0.83 \cdot 10^{-3}/r_0 =$ 3 5 5 5 $1.75 \cdot 10^{-4}/r_0 =$ 4 5 7 7 $3.47 \cdot 10^{-3}/r_0 =$	$1.1 \cdot 10^5$	110	1.1	0.011	$1.1 \cdot 10^{-5}$	
2		$0.83 \cdot 10^4$	8.3	0.083	$8.3 \cdot 10^{-4}$	$8.3 \cdot 10^{-7}$	
3		$1.75 \cdot 10^3$	1.75	0.0175	$1.75 \cdot 10^{-4}$	$1.75 \cdot 10^{-7}$	
4		$3.47 \cdot 10^4$	34.7	0.347	$3.47 \cdot 10^{-3}$	$3.47 \cdot 10^{-6}$	
5	$\sigma_{0.2}/\sigma_{cr} = 14/7$ $n_0 = 3$ $n = 5$ $\Omega = 0.1$ / $r_0 =$ 6 4 5 5 $7.6 \cdot 10^{-3}/r_0 =$ 7 5 5 5 $16 \cdot 10^{-4}/r_0 =$ 8 5 7 7 $0.0766/r_0 =$	10^6	1,000	10.	0.1	0.0001	
6		$7.6 \cdot 10^4$	76	0.76	$0.76 \cdot 10^{-2}$	$0.76 \cdot 10^{-5}$	
7		$16 \cdot 10^3$	16	0.16	$0.16 \cdot 10^{-2}$	$0.16 \cdot 10^{-5}$	
8		$766 \cdot 10^3$	766	7.66	$7.66 \cdot 10^{-2}$	$7.66 \cdot 10^{-5}$	
9	$\sigma_{cB^{(5)}}/\sigma_{0.2} = 8/9$; $n = 5$ $n_0 = 4$ $\nu = 1.36$ $\theta = 10300$ $r_0 =$ 10 5 4 431 $r_0 =$ 11 5 5 111 $r_0 =$	$1.03 \cdot 10^{-3}$	1.03	103	$1.03 \cdot 10^4$	$1.03 \cdot 10^7$	
10		$4.31 \cdot 10^{-5}$	0.0431	4.31	431	$4.31 \cdot 10^5$	
11		$1.11 \cdot 10^{-5}$	0.0111	1.11	111	$1.11 \cdot 10^5$	
12	$\sigma_{0.2}/\sigma_{cr} = 9/7$, $\sigma_{cB^{(5)}}/\sigma_{0.2} = 8/9$, $n = 5$ $n_0 = 4$ $\nu = 1.36$ $\partial_R = 62^{1/n} =$ 13 5 4 $=$ 14 5 5 $=$	$12 \cdot 10^{-3}$	1.83	52.3	$1.49 \cdot 10^3$	$2.3 \cdot 10^5$	
13		0.0170	0.0674	0.170	0.426	1.70	
14		0.0194	0.0194	0.0194	0.0194	0.0194	

Partly these discrepancies are due to the fact that influence of local contraction may have been important, particularly for higher values of ϑ_R . Other discrepancies may be accounted for utilizing the full equation (59). In his paper, Kachanov¹³ gives limits for the ratio ν/n , viz.,

$$0.62 < \frac{\nu}{n} < 1 \quad (66)$$

and suggests 0.7 as a fair mean value for the materials tested. A more critical discussion of Staniukovich's measurements is not possible as no details are given in his paper.

The table above shows the quantity ϑ_R as a function of v_0 in three typical examples. Number 12 would correspond to the case of Fig. 10—i.e., a brittle low-alloy steel. Number 13 or 14 would correspond to the 0.17 percent C-steel mentioned above.

The rupture stress would be the highest stress encountered during the test—i.e., in nondimensional form the value of Σ that occurs for $\vartheta = 0$, viz.,

$$\Sigma_R = \Omega^{-1/n} = \frac{\sigma_c}{\sigma_0} \cdot v_0^{1/n} \quad (67)$$

(B) THE CASE OF HIGH VELOCITY

We shall assume t_R to be smaller than t_1 , i.e., rupture is supposed to occur before stress has reached its maximum as determined by creep action. In such case the dimensionless stress Σ is determined by Eq. (59) only and we may write this equation in integral form satisfying the initial condition

$$\Sigma^{n_0} + \Omega \int_0^{\vartheta} \Sigma^n d\vartheta = \log(1 + \vartheta) = \vartheta - \frac{\vartheta^2}{2} + \frac{\vartheta^3}{3} \quad (68)$$

If in this equation the first term will dominate, we may find by iteration the development

$$\Sigma = \vartheta^{1/n_0} \left\{ 1 - \frac{\vartheta}{2n_0} + \frac{5n_0 + 3}{24n_0^2} \vartheta^2 - \frac{\Omega n_0 \vartheta^{n/n_0}}{n + n_0} \left[1 + \frac{\vartheta}{2n_0} \left(\frac{2n_0^2 - n^2}{n + 2n_0} - 1 \right) \dots \right] \right\} \quad (69)$$

Inserting this, in Eq. (61) we obtain

$$\theta = \frac{n_0 \vartheta_R^{r+n_0/n_0}}{n_0 + \nu} \left\{ 1 - \frac{\nu}{2n_0} \cdot \frac{n_0 + \nu}{2n_0 + \nu} \vartheta_R + \dots - \frac{\nu n \Omega}{n + n_0} \cdot \frac{(n_0 + \nu) \vartheta_R^{n/n_0}}{\nu + n_0 + n} \left[1 + \dots \right] \right\} \quad (70)$$

For small ϑ_R this equation may be solved in the form

$$\vartheta_R = \left[\frac{\theta(n_0 + \nu)}{n_0} \right]^{n_0/\nu+n_0} = \left[t_k \left(\frac{\sigma_k}{\sigma_0} \right)^\nu \left(\frac{n_0 + \nu}{n_0} \right) \right]^{n_0/n_0+\nu} \nu_0^{n_0/n_0+\nu} \quad (71)$$

and with the same degree of approximation we obtain the dimensionless rupture stress

$$\Sigma_R = \left[t_k \left(\frac{\sigma_k}{\sigma_0} \right)^\nu \left(\frac{n_0 + \nu}{n_0} \right) \right]^{1/n_0+\nu} \nu_0^{1/n_0+\nu} \quad (72)$$

Equations (71) and (72) have limited validity. It would for example not be possible to use them in the case of the steel shown in Fig. 10—i.e., about Number 12 of the table. In this case Ω would be too large. In this case Eq. (65) should offer quite satisfactory agreement.

In the case of the carbon steel represented by Number 13 in the table, it may be employed with advantage but for the smallest value of ν_0 .

It is of course difficult to make experiments to check the above theories over a large range of strain rates. I have only mentioned those of Staniukovich. In addition to these the only experiments worth mentioning, that I know of, are those of A. Nadai and M. J. Manjoine.²¹ For various reasons they hardly permit comparison with those of Staniukovich. This is mainly due to the fact that they, generally speaking, work in a higher range of strain rates than Staniukovich and only overlap occasionally in this respect. Further, Nadai and Manjoine give results principally in terms of ultimate stress and Staniukovich in terms of elongation. The only materials that are common to both investigations “mild steel” and “stainless steel” are not sufficiently specified by Nadai and Manjoine as to permit a detailed comparison. The number of readings of these investigators is small and the scatter too large, naturally as a consequence of experimental difficulties. Still the general trend is the same as in Staniukovich’s paper (see Figs. 14 and 15). When replotted to double logarithmic scale, Nadai and Man-

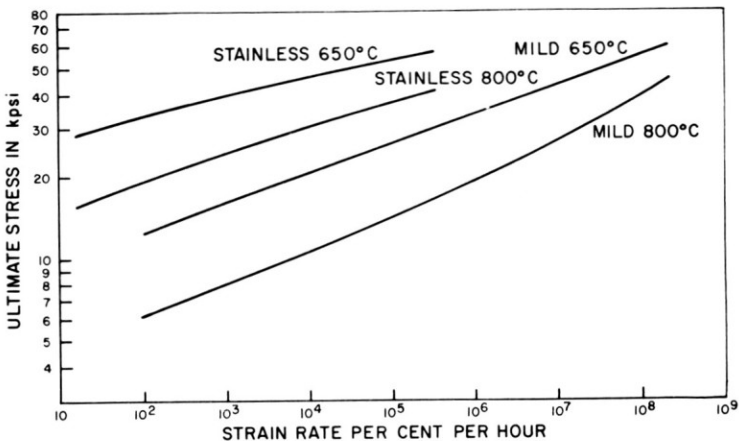


Fig. 15.

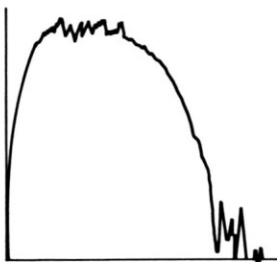
joine's values fall on straight lines just as Staniukovich's. Also, the global appearance of the stress as function of time with an early maximum as expressed in our Fig. 13 is generally in agreement with their measurements. Figure 16 (upper part) shows the stress variation with time as recorded by Nadai and Manjoine for "mild steel." The lower diagram shows for comparison result of exact integration of Eqs. (59) through (61) in the case $n_0 = n = 5$, corresponding to a 0.17 percent cast C-steel at 455°C. The early stress maximum found in the experiments, does not agree with theory in this case. I will finish by pointing to this field for future research.

A further treatment of the system of Eqs. (59) through (61) should be performed in this connection, permitting accurate comparison while using realistic values of the material constants involved. In the general case integration has to be performed numerically. The special case $n = n_0$, as just mentioned, permits analytical treatment for arbitrary value of Ω . Results of such calculations shall be published elsewhere.

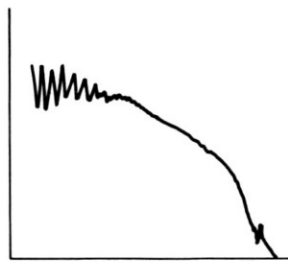
LIST OF SYMBOLS

t	= time
t^*	= rupture time, general; t_R = rupture time, according to Kachanov
t_k	= rupture time under constant stress σ_k , according to Kachanov

MILD STEEL (NADAI, MANJOINE)

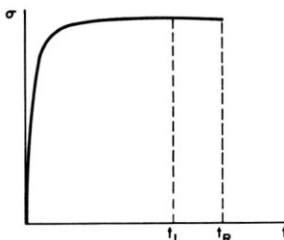


AT ROOM TEMPERATURE; STRAIN RATE, 300 PER SEC



AT 700°C; STRAIN RATE, 300 PER SEC

0.17C - CAST STEEL (THEORY)



AT 455°C, STRAIN RATE, 10^{-5} PER HOUR

$\sigma_{0.2} = 16.5$ KILOGRAMMES PER SQ. MM

$\sigma_{c.7} = 5.5$ KILOGRAMMES PER SQ. MM

$\sigma_{cB}^{(5)} = 4.0$ KILOGRAMMES PER SQ. MM

Fig. 16.

t_H^*	= rupture time, according to Hoff (ductile fracture)
t_K^*	= rupture time, according to Kachanov's theory of combined ductile and brittle fracture
t_p^*	= rupture time for ductile fracture, when taking primary creep into account
t_{Kp}^*	= rupture time for combined ductile and brittle fracture, when taking primary creep into account
t_1	= time to stress maximum in test with constant speed
ϵ	= strain, plastic
$\epsilon^{(0)}$	= value of ϵ for $t = 0$
σ	= stress
$\epsilon_1, \epsilon_2, \epsilon_3$	= principal strains
$\sigma_1, \sigma_2, \sigma_3$	= principal stresses; also $\sigma_1 =$ maximum stress in test with constant speed
σ_{10}	= value of σ_1 for $t = 0$
$\sigma_0, \sigma_{0.2}$	= proof stress
σ_c, σ_{c7}	= limiting creep stress
$\sigma_{cB}^{(5)}$	= rupture stress for $10^5 h$ life
σ_e	= effective stress
$\bar{\sigma}, \bar{\sigma}_p$	= boundary stress between ductile and brittle creep rupture
σ_r	= stress on supporting area A_r
n	= exponent of Norton's power law
ν, C	= constants of Kachanov's theory
A	= sectional area of test piece in uniaxial tension
A_0	= value of A for $t = 0$
l	= gage length
l_0	= value of l for $t = 0$
A_r	= supporting area
$D = (A - A_r)/A$	= damage factor
v	= strain rate, creep rate
v_1, v_2, v_3	= principal creep rates
v_0	= value of v at $t = 0$
P	= total load of test specimen in uniaxial tension
$\alpha = A/A_0$	= parameter
α_R	= value of α at rupture
ϑ	= $v_0 t$
ϑ_R	= $v_0 t_R$
Σ	= σ/σ_0
Ω, θ	= dimensionless quantities defined by Eq. (58)
f, F, G	= functions

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DISCUSSION

Author: F. K. G. Odqvist

Discussor: H. P. van Leeuwen, National Aero- and Astronautical Research Institute

I would like to start by congratulating Professor Odqvist on his very fine paper giving us a clear picture of the creep rupture problem and the possible way of solving it. The comment I would like to make is that in case of ductile creep (Hoff's equations), rupture is assumed to occur when the cross sectional area A reduces to zero.

Likewise Kachanov assumes brittle creep rupture to occur when the residual area A_r (the effective load bearing area in the damage expression $D = (A - A_r)/A$) reduces to zero.

In both cases the true stress goes to infinity which, I think, is a rather unrealistic situation. I wonder whether the theory could be improved somewhat by assuming rupture to occur at a finite value of the cross-sectional area A or A_r , which then should be related to the true ultimate tensile strength of the material at the test temperature, taking into account, if necessary, the embrittling effect of time-at-temperature.

As regards Kachanov's damage concept I would like to indicate the similarity with Machlin's approach published in the *Journal of Metals* some years ago, under the title "Creep Rupture by Vacancy Condensation." Machlin assumes vacancies to be produced by dislocation interaction. The vacancies coalesce into voids, reducing the actual load bearing area, a process leading to rupture ultimately.

Now Machlin was able to calculate the elongation at rupture as a function of the metal structure and shows that the product of (minimum) creep rate and rupture time would tend to be a constant.

Checking Machlin's rule with actual test data shows that it holds for pure metals having a stable structure, whereas with cold-deformed and precipitation hardened materials a clear cut dependence of the above product on temperature and time is observed.

Yet I think his approach is valuable because it will enable rupture life to be calculated as the necessary time to reach a certain critical elongation.

Author: F. K. G. Odqvist

Discussor: N. J. Hoff, Stanford University

I would like to congratulate the author on his comprehensive lecture. Since it seems to be customary at this Congress to make long-range predictions, I would like to predict that this lecture will become the definitive paper on creep rupture until and unless solid-state physicists and metallurgists put their findings in a form suitable for analytical treatment by the methods of continuum mechanics. I wonder how the author feels about the generality of the damage function proposed by Professor Kachanov. It seems to me that its validity might be restricted to tension tests; it certainly cannot be valid for hydrostatic compression because cracks cannot form in the same way under this loading.

I would like to close my remarks by expressing my satisfaction with the care with which Professor Odqvist defined the problems he treated. In the past all too often the boundary conditions of a problem were stated by the analyst in a manner that was convenient for solution, which the experimentalist built his test equipment solely for the convenience of testing. Naturally good agreement between theory and experiment can be expected only if the analyst defines the problem in a manner suitable for experimental verification; this was fully accomplished by Professor Odqvist.

Author's reply to discussion:

The improvement proposed by Mr. van Leeuwen, taking account of a finite value of the cross-sectional area at rupture has in fact been suggested by professor Hoff in his quoted paper. It has the obvious disadvantage of introducing at least one more empirical constant into the theory and will of course render the formulae more complicated.

The author is grateful for reference to professor Machlin's paper and shall have to study his approach more closely.

Professor Hoff raises the question about the validity of Kachanov's theory in the case of superimposed hydrostatic pressure. In fact Kachanov assumes the presence of at least one tensile principal stress. It should be possible to construct a crucial test for proof or disproof of this hypothesis in the following way. If a thin-walled tube is subject to torsion in the presence of hydrostatic pressure the maximum tensile principal stress could be lowered in an arbitrary way. Extrapolation to zero maximum principal stress would enable desired conclusion.